

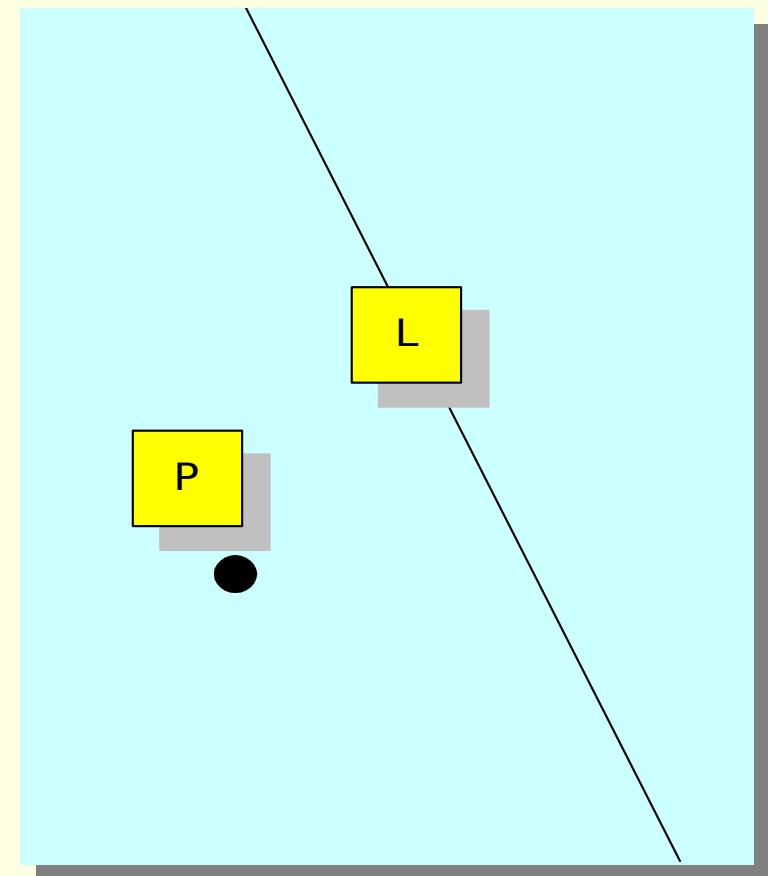
Basic Homogeneous Geometry

The Problems

2DH Points and Lines

$$\mathbf{P} = [x \ y \ w]$$

$$\mathbf{L} = \begin{matrix} \hat{e}_a \\ \hat{e}_b \\ \hat{e}_c \end{matrix} \begin{matrix} \hat{u}_a \\ \hat{u}_b \\ \hat{u}_c \end{matrix}$$

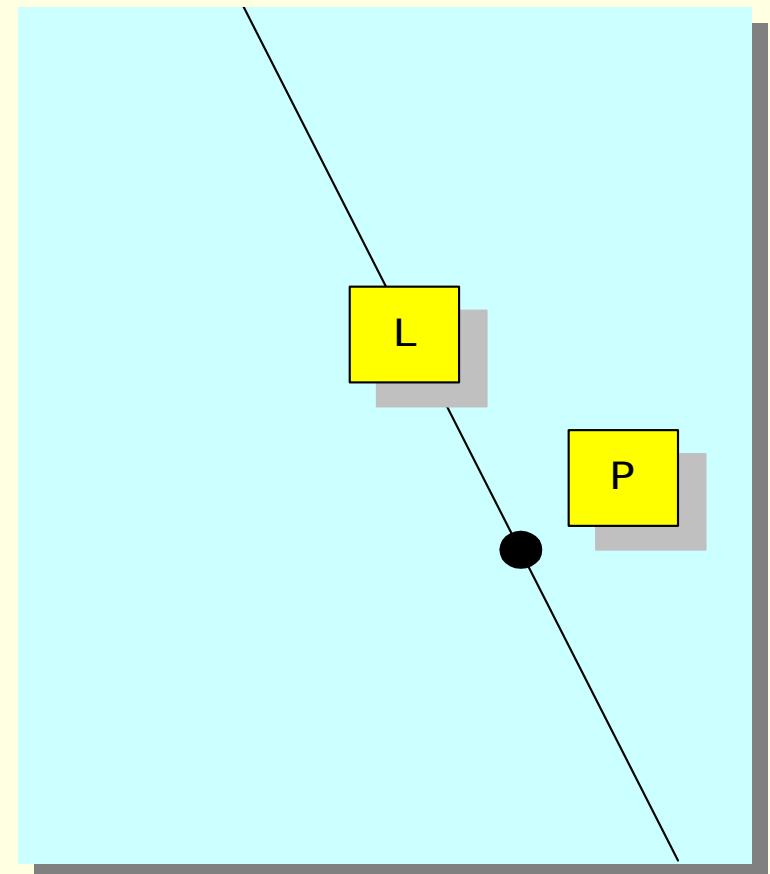


2DH Point on a Line

$$ax + by + cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\mathbf{P} \times \mathbf{L} = 0$$



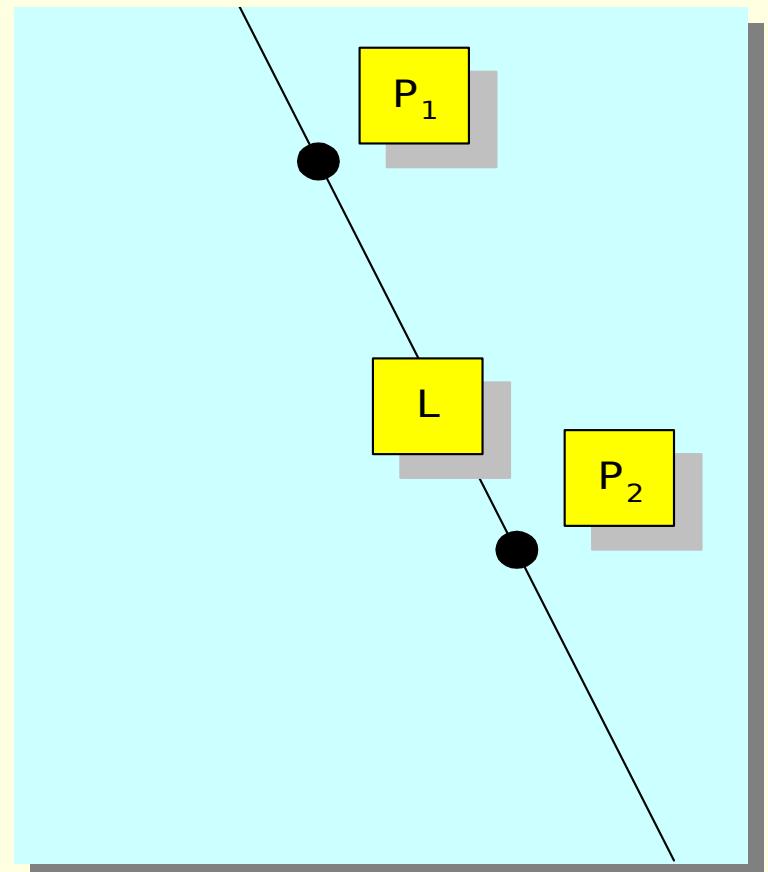
2DH Two Points Make A Line

$$\mathbf{P}_1, \mathbf{P}_2 = \mathbf{L}$$

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_a \\ \hat{e}_b \\ \hat{e}_c \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}$$

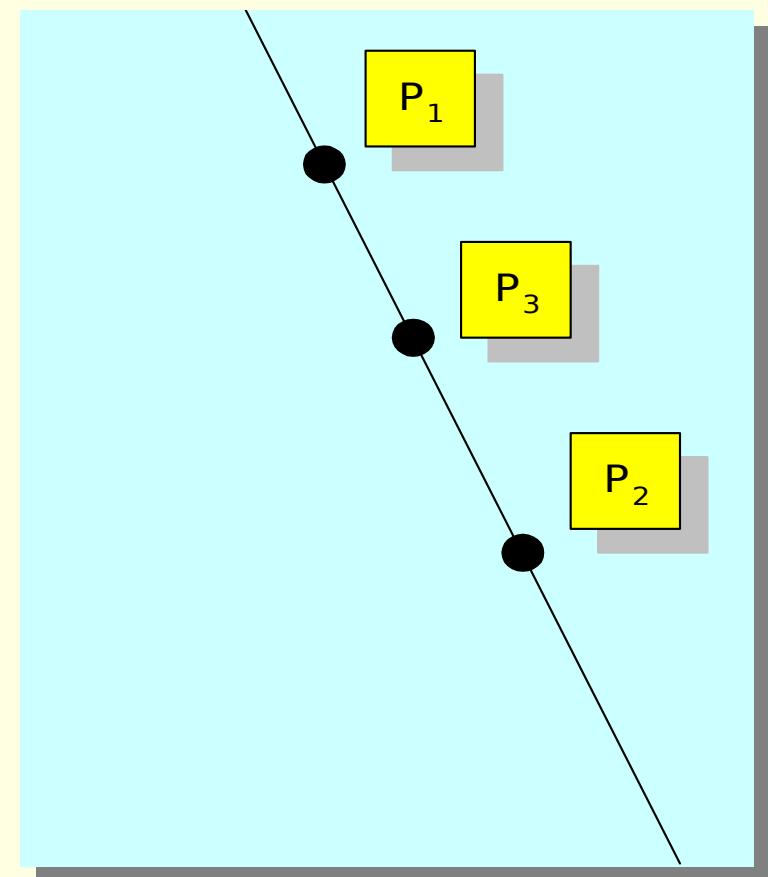
$$a = \det \begin{bmatrix} \hat{e}_y_1 & \hat{w}_1 \\ \hat{e}_y_2 & \hat{w}_2 \end{bmatrix}$$

$$b = \det \begin{bmatrix} \hat{e}_w_1 & x_1 \\ \hat{e}_w_2 & x_2 \end{bmatrix} \quad c = \det \begin{bmatrix} \hat{e}_x_1 & y_1 \\ \hat{e}_x_2 & y_2 \end{bmatrix}$$



2DH Three Collinear Points

$$\mathbf{P}_1 \cdot \mathbf{P}_2 \times \mathbf{P}_3 = 0$$



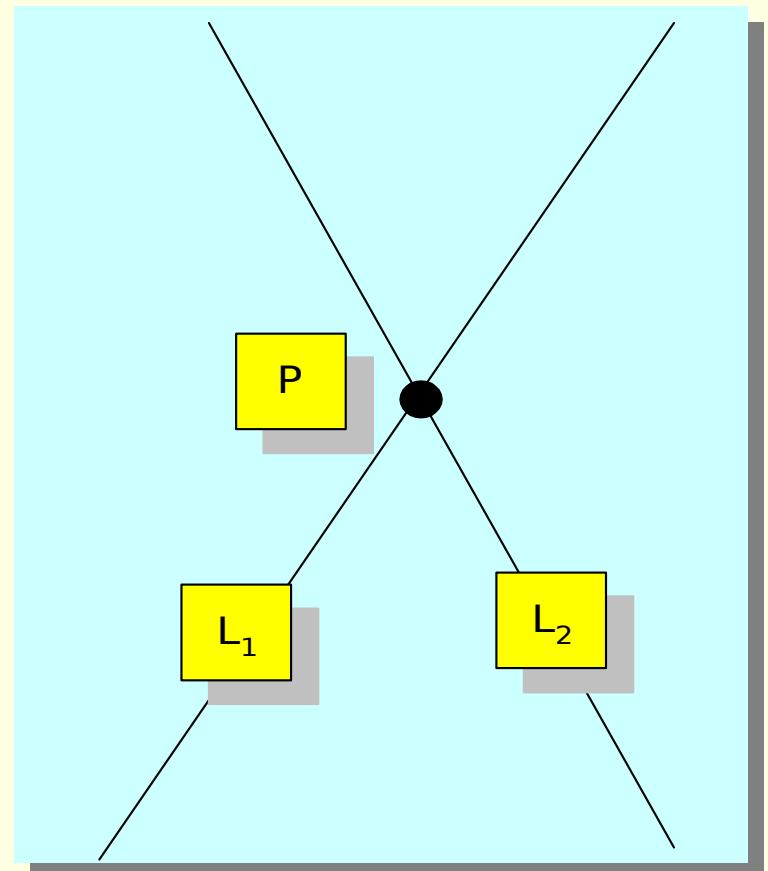
2DH Two Lines Make A Point

$L_1, L_2 = P$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \hat{e}_1 & \hat{e}_2 & \hat{e} \end{vmatrix} = [x \ y \ w]$$

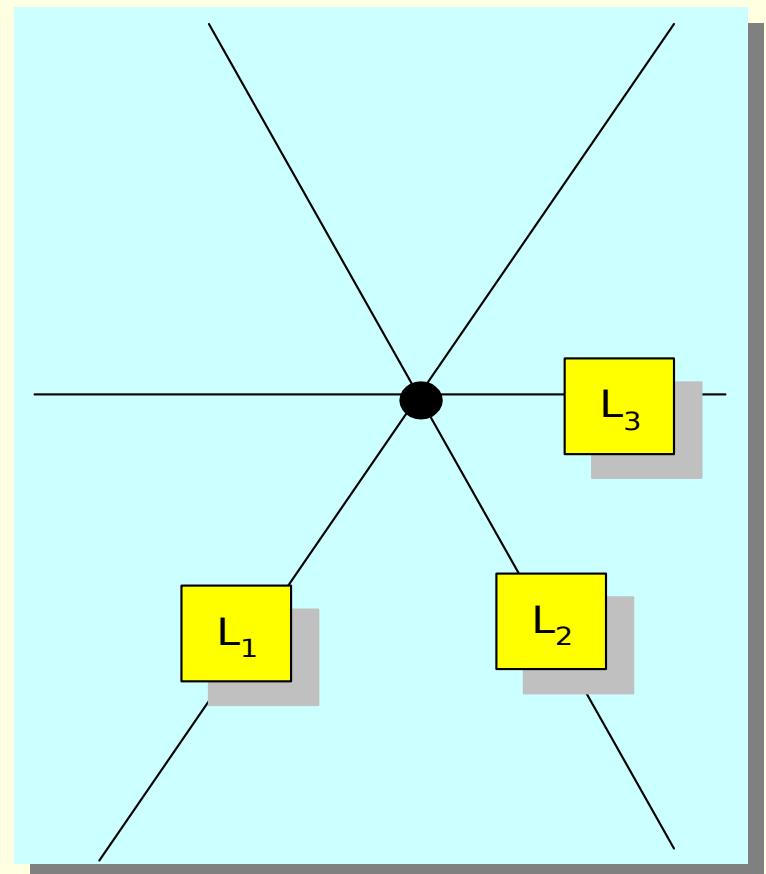
$$x = \det \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad y = \det \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$z = \det \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$



2DH Three CoPointar Lines

$$\mathbf{L}_1 \cdot \mathbf{L}_2 \times \mathbf{L}_3 = 0$$



2DH Transforming Points

$$\mathbf{PT} = \hat{\mathbf{P}}$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{w} \end{bmatrix}$$

2DH Transforming Lines

$$\mathbf{P} \mathbf{L} = 0 \quad \hat{\mathbf{P}} \hat{\mathbf{L}} = 0$$

$$\begin{aligned}\mathbf{P} \mathbf{L} &= \mathbf{P} \left(\mathbf{T} \mathbf{T}^{-1} \right) \mathbf{L} \\ &= (\mathbf{P} \mathbf{T}) \left(\mathbf{T}^{-1} \mathbf{L} \right) \\ &= \hat{\mathbf{P}} \left(\mathbf{T}^{-1} \mathbf{L} \right)\end{aligned}$$

$$\mathbf{T}^{-1} \mathbf{L} = \hat{\mathbf{L}}$$

2DH Matrix Adjoint

$$\mathbf{T} = \begin{bmatrix} R_1 & M \\ R_2 & M \\ R_3 & M \end{bmatrix}$$

$$\mathbf{T}^* = \begin{bmatrix} M & M & M \\ R_2' & R_3' & R_1' \\ M & M & M \end{bmatrix}$$

$$\mathbf{T}\mathbf{T}^* = (\det \mathbf{T}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}^* = (\det \mathbf{T}) \mathbf{T}^{-1}$$

2DH Transforming Points and Lines

$$\mathbf{P}\mathbf{T} = \hat{\mathbf{P}}$$
$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \hat{e}T_{11} & T_{12} & T_{13} \\ \hat{e}T_{21} & T_{22} & T_{23} \\ \hat{e}T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{w} \end{bmatrix}$$

$$\mathbf{T}^* \mathbf{L} = \hat{\mathbf{L}}$$
$$\begin{bmatrix} \hat{e}T_{11}^* & T_{12}^* & T_{13}^* \\ \hat{e}T_{21}^* & T_{22}^* & T_{23}^* \\ \hat{e}T_{31}^* & T_{32}^* & T_{33}^* \end{bmatrix} \begin{bmatrix} \hat{u} & \hat{a} & \hat{u} \\ \hat{u} & \hat{b} & \hat{u} \\ \hat{u} & \hat{c} & \hat{u} \end{bmatrix} = \begin{bmatrix} \hat{e}u & \hat{a}u & \hat{u} \\ \hat{e}u & \hat{b}u & \hat{u} \\ \hat{e}u & \hat{c}u & \hat{u} \end{bmatrix}$$

2DH Point on Quadratic Curve

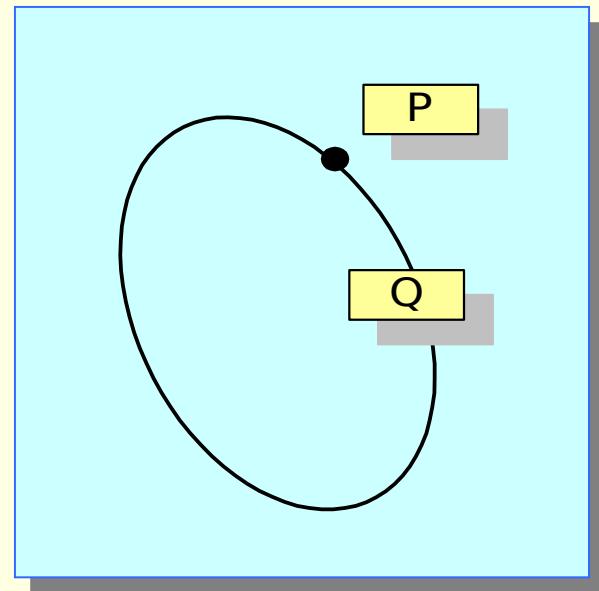
$$Ax^2 + 2Bxy + 2Cxw$$

$$+Dy^2 + 2Eyw$$

$$+Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$\mathbf{P} \times \mathbf{Q} \times \mathbf{P}^T = 0$$



2DH Transforming a Quadratic

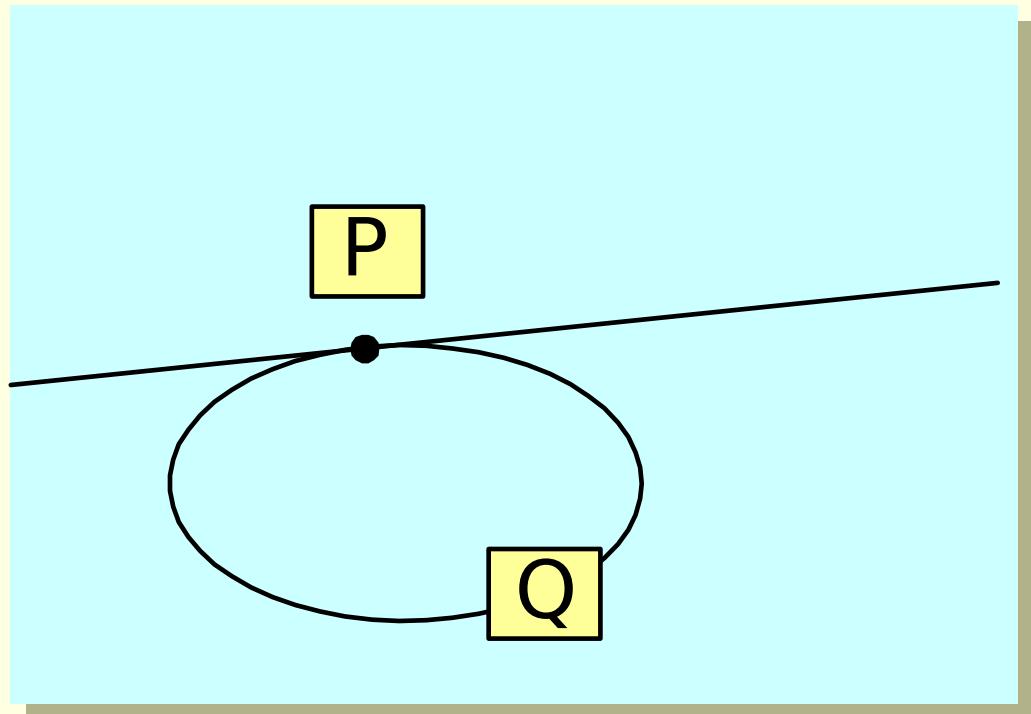
$$\mathbf{PQ}\mathbf{P}^T = 0 \quad \hat{\mathbf{U}} \quad \hat{\mathbf{P}}\hat{\mathbf{Q}}\hat{\mathbf{P}}^T = 0$$

$$\begin{aligned}\mathbf{PQ}\mathbf{P}^T &= d^2 \quad \mathbf{P} \left(\mathbf{T}\mathbf{T}^* \right) \mathbf{Q} \left(\mathbf{T}\mathbf{T}^* \right)^T \mathbf{P}^T \\ &= d^2 \quad (\mathbf{P}\mathbf{T}) \left(\mathbf{T}^* \mathbf{Q} \mathbf{T}^{*T} \right) (\mathbf{P}\mathbf{T})^T \\ &= d^2 \quad \hat{\mathbf{P}} \left(\mathbf{T}^* \mathbf{Q} \mathbf{T}^{*T} \right) \hat{\mathbf{P}}^T\end{aligned}$$

$$\boxed{\mathbf{T}^* \mathbf{Q} \mathbf{T}^{*T} = \hat{\mathbf{Q}}}$$

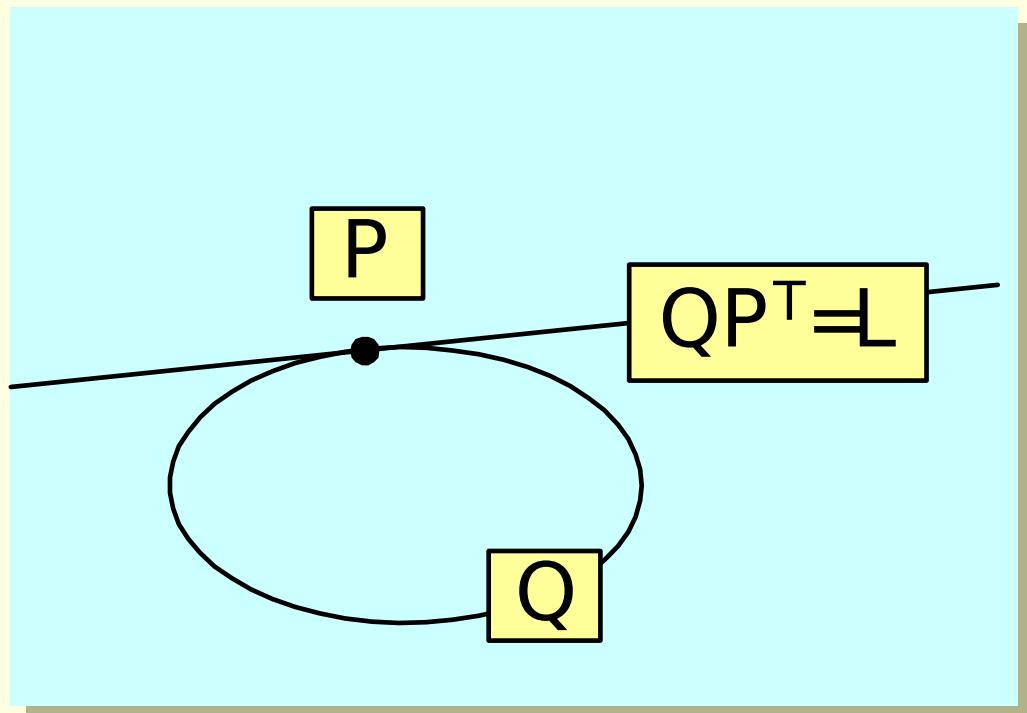
2DH Tangent at a Point

$$\begin{aligned}0 &= \mathbf{P} \mathbf{Q} \mathbf{P}^T \\&= \mathbf{P} \cancel{\times} \mathbf{Q} \mathbf{P}^T \\&= \mathbf{P} \cancel{\times} \mathbf{L}\end{aligned}$$



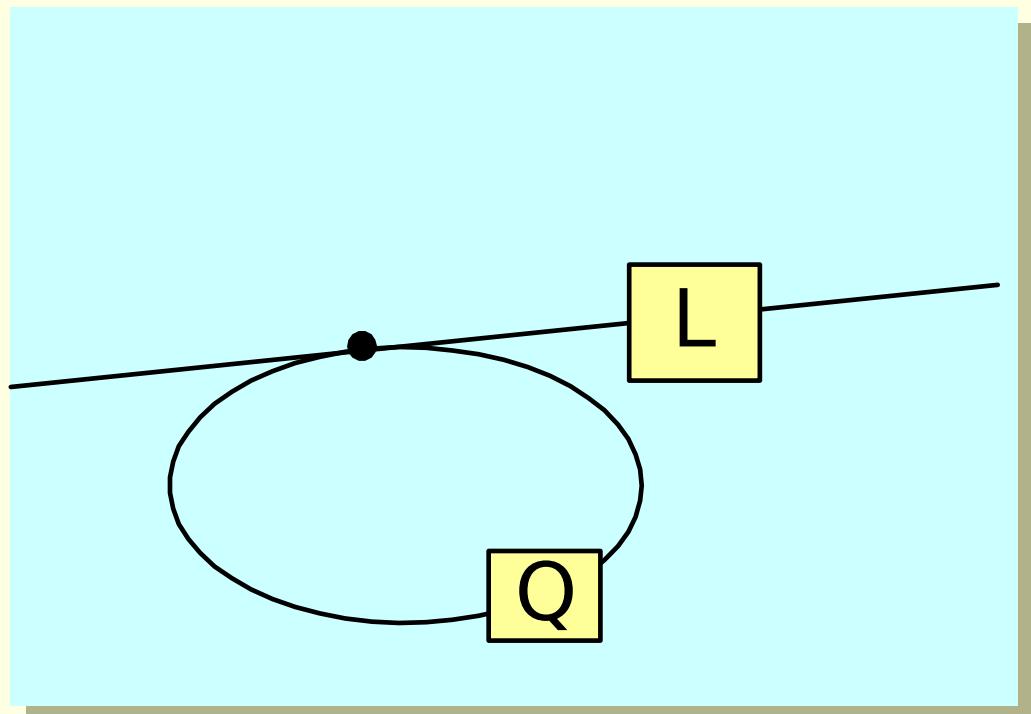
2DH Tangent at a Point

$$\begin{aligned}0 &= \mathbf{P} \mathbf{Q} \mathbf{P}^T \\&= \mathbf{P} \cancel{\times} \mathbf{Q} \mathbf{P}^T \\&= \mathbf{P} \cancel{\times} \mathbf{L}\end{aligned}$$



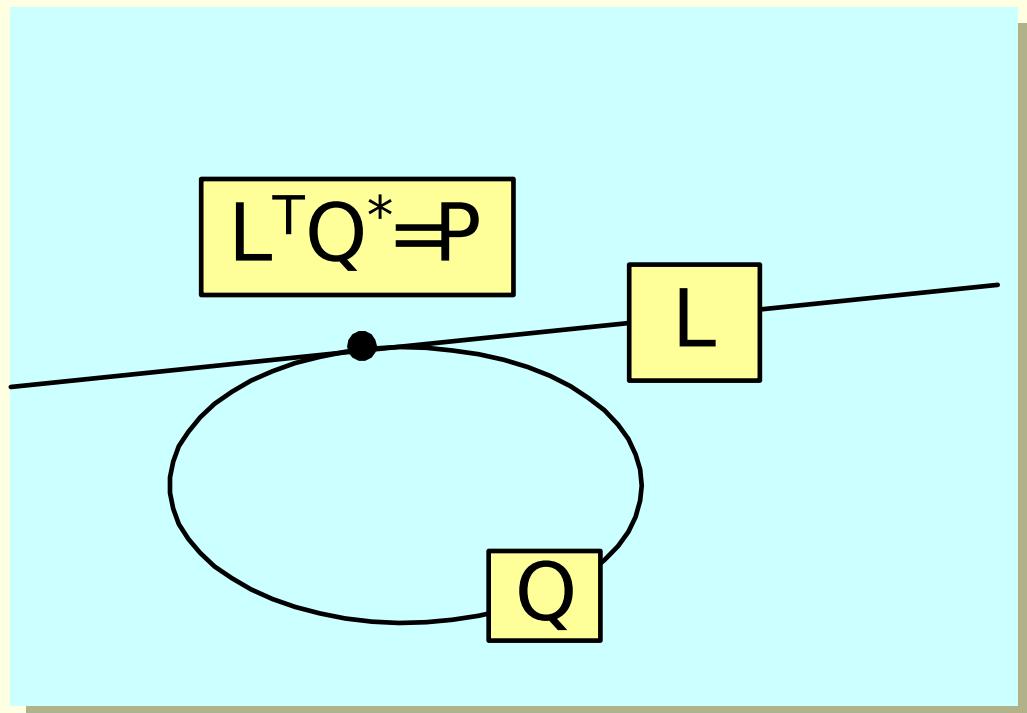
2DH Line Tangent to Quadric

$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$
$$= (\mathbf{L}^T \mathbf{Q}^*) \mathbf{L}$$



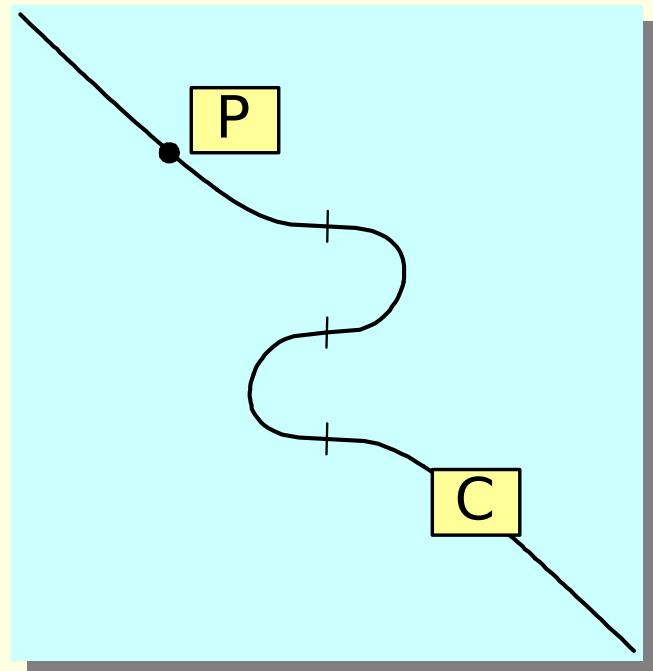
2DH Line Tangent to Quadric

$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$
$$= (\mathbf{L}^T \mathbf{Q}^*) \mathbf{L}$$



2DH Point on Cubic Curve

$$\begin{aligned} & Ax^2 + 3Bx^2y + 3Cxy^2 + Dy^3 \\ & + 3Ex^2w + 6Fxyw + 3Gyw^2 \\ & + 2Hxw^2 + 3Jyw^2 \\ & + Kw^3 = 0 \end{aligned}$$



2DH Cubic Curve

$$\begin{aligned} & Ax^2 + 3Bx^2y + 3Cxy^2 + Dy^3 \\ & + 3Ex^2w + 6Fxyw + 3Gyw^2 \\ & + 2Hxw^2 + 3Jyw^2 \\ & + Kw^3 = 0 \end{aligned}$$

$$\begin{bmatrix} 1 & x & y & w \end{bmatrix} \begin{bmatrix} A & B & C & D & E & F & G & H & I & J & K \\ B & C & D & E & F & G & H & I & J & K & L \\ C & D & E & F & G & H & I & J & K & L & M \\ D & E & F & G & H & I & J & K & L & M & N \\ E & F & G & H & I & J & K & L & M & N & O \\ F & G & H & I & J & K & L & M & N & O & P \\ G & H & I & J & K & L & M & N & O & P & Q \\ H & I & J & K & L & M & N & O & P & Q & R \\ I & J & K & L & M & N & O & P & Q & R & S \\ J & K & L & M & N & O & P & Q & R & S & T \\ K & L & M & N & O & P & Q & R & S & T & U \\ L & M & N & O & P & Q & R & S & T & U & V \\ M & N & O & P & Q & R & S & T & U & V & W \\ N & O & P & Q & R & S & T & U & V & W & X \\ O & P & Q & R & S & T & U & V & W & X & Y \\ P & Q & R & S & T & U & V & W & X & Y & Z \end{bmatrix} = 0$$

$$\left\{ \mathbf{P} \mathbf{C} \mathbf{P}^T \right\} \mathbf{P}^T = 0$$

2DH Curves of Various Orders

$$L = \begin{matrix} \hat{e}^a \\ \hat{e}^b \\ \hat{e}^c \end{matrix} \begin{matrix} \hat{u} \\ \hat{u} \\ \hat{u} \end{matrix}$$

$$Q = \begin{matrix} \hat{e}^A \\ \hat{e}^B \\ \hat{e}^C \end{matrix} \begin{matrix} A \\ B \\ C \end{matrix} \begin{matrix} \hat{u} \\ \hat{u} \\ \hat{u} \end{matrix}$$

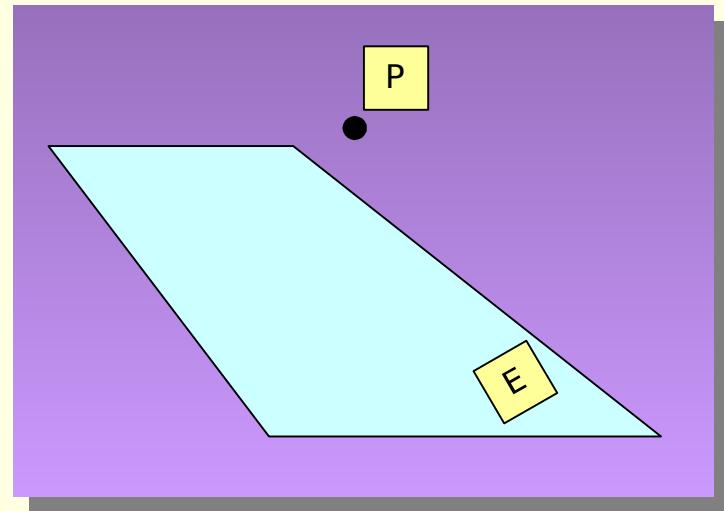
$$C = \begin{matrix} \hat{e}^e \\ \hat{e}^e \\ \hat{e}^e \\ \hat{e}^e \end{matrix} \begin{matrix} \hat{e}^A \\ \hat{e}^B \\ \hat{e}^C \\ \hat{e}^D \end{matrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{matrix} \hat{u} \\ \hat{u} \\ \hat{u} \\ \hat{u} \end{matrix} \begin{matrix} \hat{e}^F \\ \hat{e}^F \\ \hat{e}^F \\ \hat{e}^F \end{matrix} \begin{matrix} F \\ G \\ H \\ I \end{matrix} \begin{matrix} \hat{u} \\ \hat{u} \\ \hat{u} \\ \hat{u} \end{matrix} \begin{matrix} \hat{e}^E \\ \hat{e}^E \\ \hat{e}^E \\ \hat{e}^E \end{matrix} \begin{matrix} E \\ F \\ G \\ H \end{matrix} \begin{matrix} \hat{u} \\ \hat{u} \\ \hat{u} \\ \hat{u} \end{matrix} \begin{matrix} \hat{e}^H \\ \hat{e}^H \\ \hat{e}^H \\ \hat{e}^H \end{matrix} \begin{matrix} H \\ I \\ J \\ K \end{matrix} \begin{matrix} \hat{u} \\ \hat{u} \\ \hat{u} \\ \hat{u} \end{matrix}$$

Now 3D (Homogeneous)

3DH Points and Planes

$$P = [x \ y \ z \ w]$$

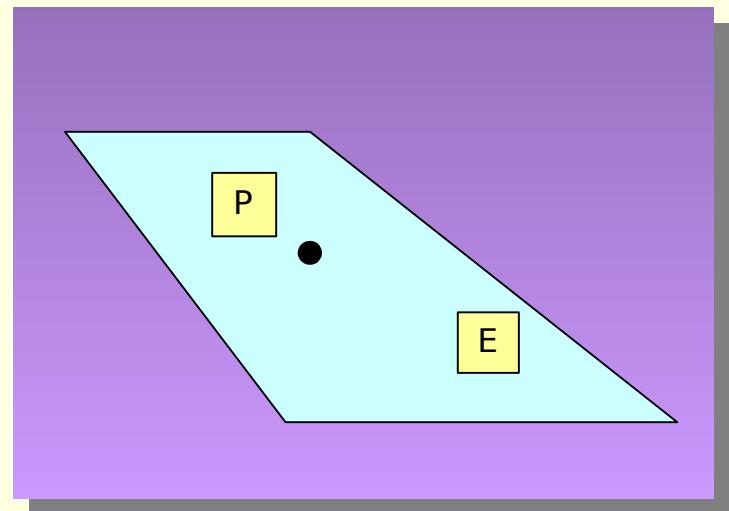
$$E = \begin{bmatrix} \hat{e}_a \\ \hat{e}_b \\ \hat{e}_c \\ \hat{e}_d \end{bmatrix}$$



3DH Point on Plane

$$[x \ y \ z \ w] \begin{bmatrix} \hat{e}_A \\ \hat{e}_B \\ \hat{e}_C \\ \hat{e}_D \end{bmatrix} = 0$$

$$\mathbf{P} \times \mathbf{E} = 0$$



3DH Transformations

$$\mathbf{PT} = \mathbf{P}\mathbf{\hat{C}}$$

$$\mathbf{T}^*\mathbf{E} = \mathbf{E}\mathbf{\hat{C}}$$

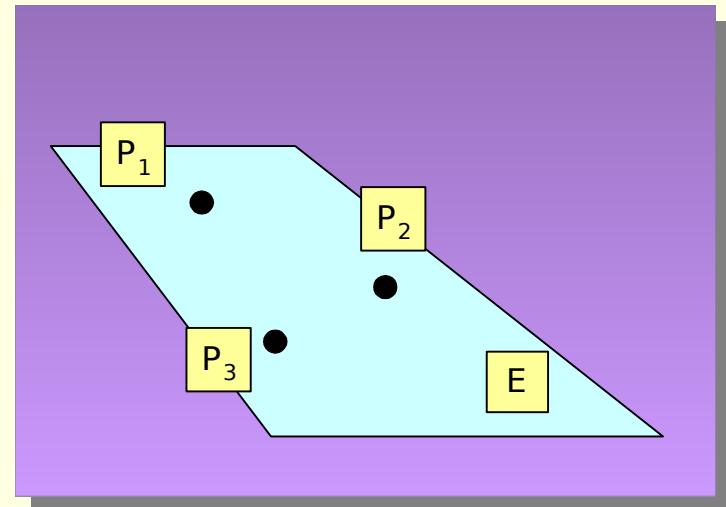
3DH Plane thru 3 Points

$$\text{cross}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3) = \mathbf{E}$$

$$\text{cross} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \end{bmatrix}, \text{ö} \begin{bmatrix} x_2 & y_2 & z_2 & w_2 \end{bmatrix}, \div \begin{bmatrix} x_3 & y_3 & z_3 & w_3 \end{bmatrix} \div \begin{bmatrix} \hat{e}_a & \hat{e}_b & \hat{e}_c & \hat{e}_d \end{bmatrix}$$

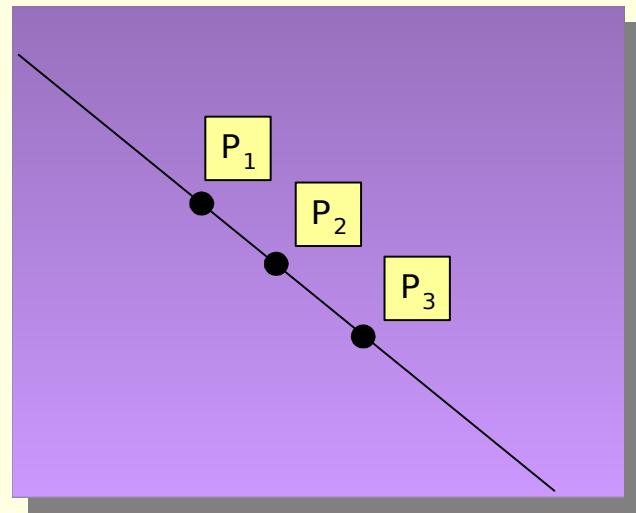
$$a = \det \begin{bmatrix} \hat{e}_y_1 & z_1 & w_1 \\ \hat{e}_y_2 & z_2 & w_2 \\ \hat{e}_y_3 & z_3 & w_3 \end{bmatrix} \quad b = - \det \begin{bmatrix} \hat{e}_x_1 & z_1 & w_1 \\ \hat{e}_x_2 & z_2 & w_2 \\ \hat{e}_x_3 & z_3 & w_3 \end{bmatrix}$$

$$c = \det \begin{bmatrix} \hat{e}_y_1 & y_1 & w_1 \\ \hat{e}_y_2 & y_2 & w_2 \\ \hat{e}_y_3 & y_3 & w_3 \end{bmatrix} \quad d = - \det \begin{bmatrix} \hat{e}_x_1 & y_1 & z_1 \\ \hat{e}_x_2 & y_2 & z_2 \\ \hat{e}_x_3 & y_3 & z_3 \end{bmatrix}$$



3DH Three Collinear points

cross $\begin{bmatrix} x_1 & y_1 & z_1 & w_1 \end{bmatrix}, \begin{bmatrix} x_2 & y_2 & z_2 & w_2 \end{bmatrix}, \begin{bmatrix} x_3 & y_3 & z_3 & w_3 \end{bmatrix} \stackrel{\div}{=} \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$



3DH Rewrite Equation

$$\text{cross} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \end{bmatrix}, \begin{bmatrix} x_2 & y_2 & z_2 & w_2 \end{bmatrix}, \begin{bmatrix} x_3 & y_3 & z_3 & w_3 \end{bmatrix} \stackrel{\text{det}}{=} 0$$

$$0 = \det \begin{bmatrix} \hat{e}y_1 & z_1 & w_1 \\ \hat{e}y_2 & z_2 & w_2 \\ \hat{e}y_3 & z_3 & w_3 \end{bmatrix}$$

$$0 = y_3 \det \begin{bmatrix} \hat{e}z_1 & w_1 \\ \hat{e}z_2 & w_2 \end{bmatrix} - z_3 \det \begin{bmatrix} \hat{e}y_1 & w_1 \\ \hat{e}y_2 & w_2 \end{bmatrix} + w_3 \det \begin{bmatrix} \hat{e}y_1 & z_1 \\ \hat{e}y_2 & z_2 \end{bmatrix}$$

$$0 = \hat{e}_0 \det \begin{bmatrix} \hat{e}z_1 & w_1 \\ \hat{e}z_2 & w_2 \end{bmatrix} - \det \begin{bmatrix} \hat{e}y_1 & w_1 \\ \hat{e}y_2 & w_2 \end{bmatrix} \det \begin{bmatrix} \hat{e}y_1 & z_1 \\ \hat{e}y_2 & z_2 \end{bmatrix} + \hat{e}w_3 \det \begin{bmatrix} \hat{e}y_1 & z_1 \\ \hat{e}y_2 & z_2 \end{bmatrix}$$

3DH Separate $P_1 P_2$ from P_3

$$\begin{array}{cccccc}
 \hat{e} & 0 & p & - q & r \hat{u} \hat{e} x_3 \hat{u} & \hat{e} 0 \hat{u} \\
 \hat{e} & p & 0 & s & - t \hat{u} \hat{e} y_3 \hat{u} & \hat{e} 0 \hat{u} \\
 \hat{e} & q & - s & 0 & u \hat{u} \hat{e} z_3 \hat{u} & \hat{e} 0 \hat{u} \\
 \hat{e} & - r & t & - u & 0 \hat{u} \hat{e} w_3 \hat{u} & \hat{e} 0 \hat{u}
 \end{array}$$

$$p = \det \begin{vmatrix} \hat{e} z_1 & w_1 \hat{u} \\ \hat{e} z_2 & w_2 \hat{u} \end{vmatrix}$$

$$q = \det \begin{vmatrix} \hat{e} y_1 & w_1 \hat{u} \\ \hat{e} y_2 & w_2 \hat{u} \end{vmatrix}$$

$$r = \det \begin{vmatrix} \hat{e} y_1 & z_1 \hat{u} \\ \hat{e} y_2 & z_2 \hat{u} \end{vmatrix}$$

$$s = \det \begin{vmatrix} \hat{e} x_1 & w_1 \hat{u} \\ \hat{e} x_2 & w_2 \hat{u} \end{vmatrix}$$

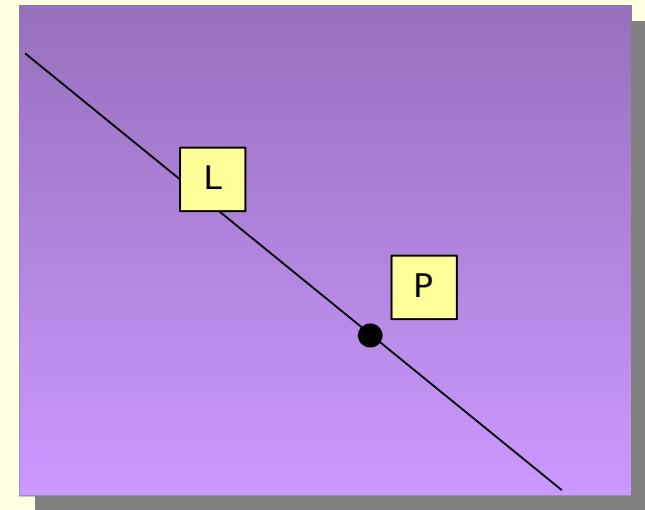
$$t = \det \begin{vmatrix} \hat{e} x_1 & z_1 \hat{u} \\ \hat{e} x_2 & z_2 \hat{u} \end{vmatrix}$$

$$u = \det \begin{vmatrix} \hat{e} x_1 & y_1 \hat{u} \\ \hat{e} x_2 & y_2 \hat{u} \end{vmatrix}$$

$$pu - qt + sr = 0$$

3DH Point on Line

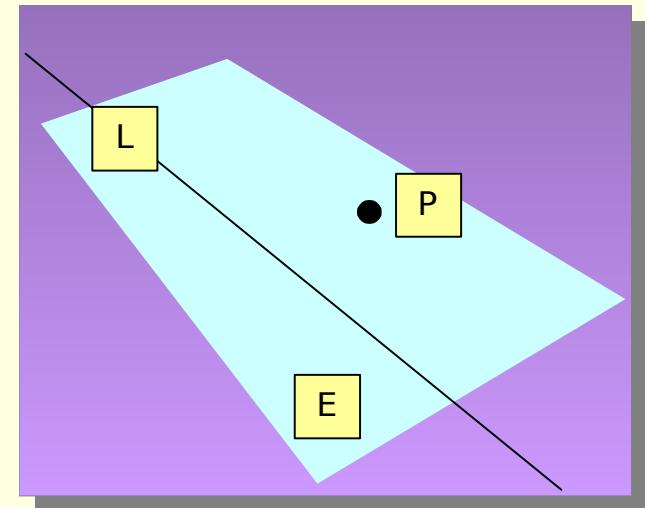
$$\begin{array}{cccccc} \hat{e}_x & 0 & p & - q & r \hat{e}_x \hat{e}_x & \hat{e}_0 \hat{e}_0 \\ \hat{e}_y & p & 0 & s & - t \hat{e}_y \hat{e}_y & \hat{e}_0 \hat{e}_0 \\ \hat{e}_z & q & - s & 0 & u \hat{e}_z \hat{e}_z & \hat{e}_0 \hat{e}_0 \\ \hat{e}_w & -r & t & -u & 0 \hat{e}_w \hat{e}_w & \hat{e}_0 \hat{e}_0 \end{array}$$



$$\mathbf{L} \mathbf{P}^T = 0$$

3DH Point not on Line = Plane

$$\begin{matrix} \mathbf{e} & 0 & p & -q & r & u & x & u & \mathbf{e} & a & u \\ \mathbf{e} & p & 0 & s & -t & u & \hat{e} & y & u & \hat{e} & b & u \\ \mathbf{e} & q & -s & 0 & u & u & \hat{e} & z & u & \hat{e} & c & u \\ \mathbf{e} & -r & t & -u & 0 & u & \hat{e} & w & u & \hat{e} & d & u \end{matrix}$$



$$\mathbf{LP}^T = \mathbf{E}$$

3DH Transforming a Line

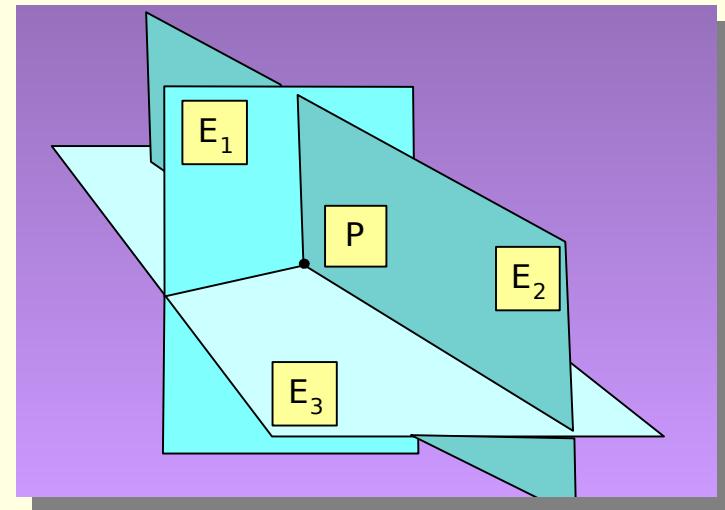
$$\mathbf{LP}^T = \mathbf{E} \hat{\mathbf{U}} \quad \mathbf{L}\Phi\mathbf{P}^T = \mathbf{E}\mathbf{C}$$

$$\mathbf{T}^* \mathbf{L} \left(\mathbf{T}^* \right)^T = \mathbf{L} \mathbf{C}$$

3DH Point on 3 Planes

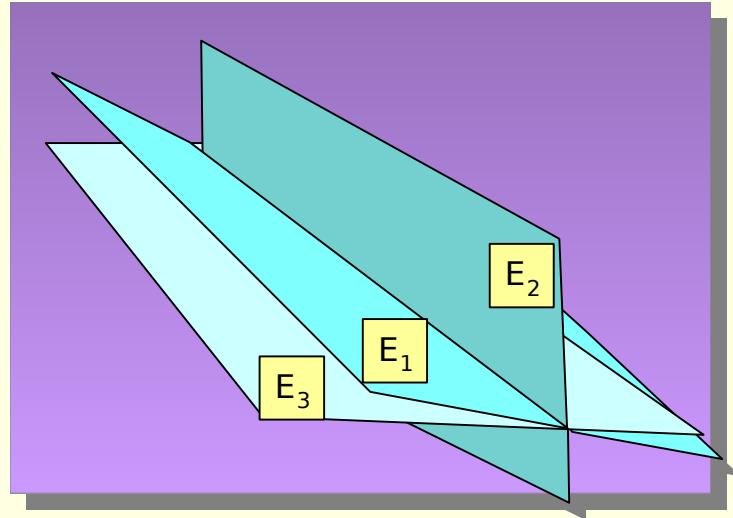
$$\text{cross}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3) = \mathbf{P}$$

$$\begin{aligned}
 & \text{det} \begin{vmatrix} a_1 & a_2 & a_3 & w \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} \\
 & \text{crs4} \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & w \\ \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & 1 \\ \hat{d}_1 & \hat{d}_2 & \hat{d}_3 & 0 \end{vmatrix} = [x \quad y \quad z \quad w] \\
 & \text{det} \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & w \\ \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & 1 \\ \hat{d}_1 & \hat{d}_2 & \hat{d}_3 & 0 \end{vmatrix}
 \end{aligned}$$



$$x = \det \begin{vmatrix} b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 0 \\ \hat{d}_1 & \hat{d}_2 & \hat{d}_3 & 0 \end{vmatrix} \quad y = - \det \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ c_1 & c_2 & c_3 & 0 \\ \hat{d}_1 & \hat{d}_2 & \hat{d}_3 & 0 \end{vmatrix} \quad z = \det \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 0 \\ \hat{d}_1 & \hat{d}_2 & \hat{d}_3 & 0 \end{vmatrix} \quad w = - \det \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 0 \\ \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & 0 \end{vmatrix}$$

3DH Three Collinear Planes



$$\begin{matrix} \text{crs4} \\ \text{a} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{e} \end{matrix} \begin{matrix} \hat{a}_1 \\ \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{matrix} \begin{matrix} \hat{a}_2 \\ \hat{b}_2 \\ \hat{b}_3 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{d}_2 \\ \hat{d}_3 \end{matrix} \begin{matrix} \hat{a}_3 \\ \hat{b}_3 \\ \hat{c}_3 \\ \hat{d}_3 \end{matrix} \begin{matrix} \hat{o} \\ \div \\ \div \\ \div \\ \emptyset \end{matrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

3DH Separate $E_1 E_2$ from E_3

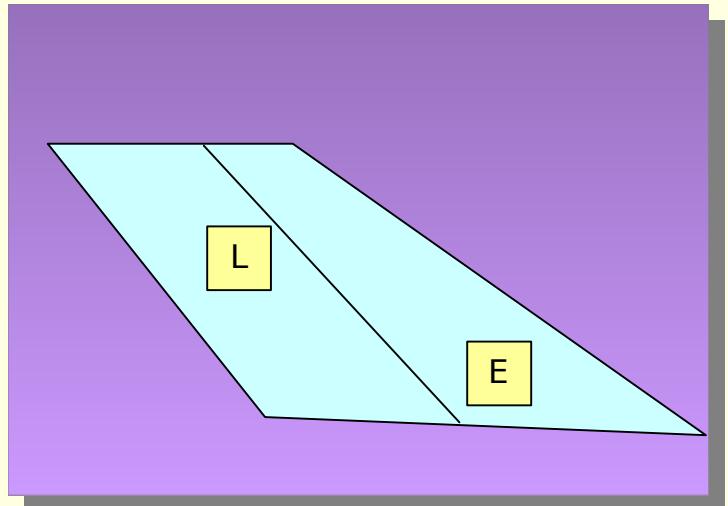
$$[a_3 \ b_3 \ c_3 \ d_3] \begin{bmatrix} e & 0 & e & -f \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ g & j & -k & 0 \end{bmatrix} = [0 \ 0 \ 0 \ 0]$$

$$e = \det \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \quad f = \det \begin{vmatrix} b_1 & b_2 \\ d_1 & d_2 \end{vmatrix} \quad g = \det \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$h = \det \begin{vmatrix} a_1 & a_2 \\ d_1 & d_2 \end{vmatrix} \quad j = \det \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \quad k = \det \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

3DH Line embedded in Plane

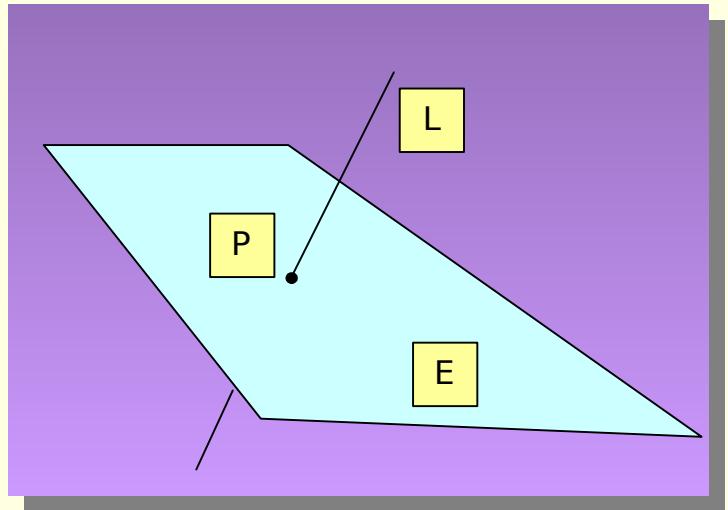
$$[a \ b \ c \ d] \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ g & j & -k & 0 \end{bmatrix} = [0 \ 0 \ 0 \ 0]$$



$$\mathbf{E}^T \mathbf{K} = \mathbf{0}$$

3DH Line Not in Plane = Point

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ g & j & -k & 0 \end{bmatrix} = \begin{bmatrix} x & y & z & w \end{bmatrix}$$



$$\mathbf{E}^T \mathbf{K} = \mathbf{P}$$

3DH Two Forms of Line

$$\begin{matrix} 0 & p & -q & r \\ p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{matrix} = \mathbf{L}$$

$$\begin{matrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ g & j & -k & 0 \end{matrix} = \mathbf{K}$$

$$\mathbf{LP}^T = \mathbf{E}$$

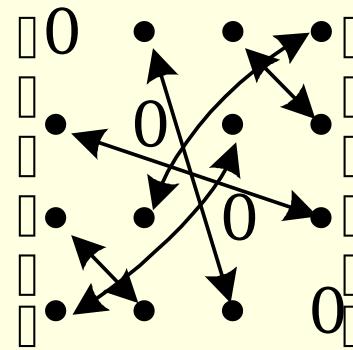
$$\mathbf{E}^T \mathbf{K} = \mathbf{P}$$

3DH Converting Between Two Forms of Line

$$\mathbf{L} = \begin{pmatrix} 0 & p & -q & r \\ p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ g & j & -k & 0 \end{pmatrix} = \mathbf{K} = \begin{pmatrix} 0 & -u & -t & -s \\ u & 0 & -r & -q \\ t & r & 0 & -p \\ s & q & p & 0 \end{pmatrix}$$

$$\begin{aligned} e &= -u, & f &= t, & g &= -s \\ h &= -r, & j &= q, & k &= -p \end{aligned}$$



Two Problems

Rows vs. Columns

$$\begin{array}{cccc} A & B & C & D \\ B & D & E & F \\ C & E & F & G \end{array} \quad \begin{array}{c} \text{Rows} \\ \text{Columns} \end{array} \quad \begin{array}{c} \text{Rows} \\ \text{Columns} \end{array}$$

$[x \ y \ z \ w] = [a \ b \ c \ d]$

$\begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ g & j & -k & 0 \end{bmatrix}$

More than Two Indices

$$\begin{array}{cccccccccc} A & B & C & D & E & F & G & H & I \\ B & C & D & E & F & G & H & I & J \\ C & D & E & F & G & H & I & J & K \\ D & E & F & G & H & I & J & K & L \end{array}$$